Basic Vector Review

A vector (also called a direction vector) is just something that has both magnitude (length, or size) and direction. So it’s different than a regular number, since it really has two components to it. We see vectors represented by arrows, so we can remember that we need to get a length of a vector (the magnitude), as well as the direction (which way it’s pointing). We use vectors in mathematics, engineering, and physics, since many times we need to know both the size of something and which way it’s going. For example, with an airplane, we can use a vector to measure the speed of the plane (the “size”) and the direction it’s flying.

**Geometric Vectors** are directed line segments in the xy-plane, and, as an example, the vector from a point \(A\) (initial point) to a point \(B\) (terminal point) can be represented by \(\overrightarrow{AB}\).

So, for example, if \(A\) is \((2, 7)\) and \(B\) is \((-3, 8)\), the vector is second point minus first point, or \(\langle x_2 - x_1, y_2 - y_1 \rangle\), or \(\langle -3 - 2, 8 - 7 \rangle = \langle -5, 1 \rangle\). The “x” part of the vector (-5) is called the x-component, and the “y” part (1) is called the y-component.

Note also that vectors can also be written in the form \(ai + bj\), so this vector can also be written as \(-5i + 1j\), or \(-5i + j\).

The **magnitude** of the vector, written \(\|AB\|\), is the distance between the two points (like the hypotenuse of a right triangle), or , or with the new vector \(\langle x, y \rangle\), it’s just \(\sqrt{x^2 + y^2}\). So for our points \(A\) and \(B\) above, \(\|AB\| = \sqrt{(-5)^2 + 1^2} = \sqrt{26}\).

Now looking at this vector visually, do you see how we can use the **slope** of the line of the vector (from the initial point to the terminal point) to get the **direction** of the vector? Pretty cool! So we can just use \(\tan^{-1}\left(\frac{y}{x}\right)\) (second part of vector over first part of vector) to get the angle measurement of the vector’s direction. Remembering from the Polar Coordinates, Equations and Graphs section though, we have to be careful which quadrant the vector terminates in (“pretending” that the vector’s initial point is at the origin) to know how many degrees we should add to that tangent value when we use a calculator:

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<tbody>
<tr>
<td>II</td>
<td>I</td>
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<tr>
<td></td>
<td>+ 180° or (\pi) (Add nothing)</td>
</tr>
<tr>
<td>+ 180° or (\pi)</td>
<td>+ 360° or 2(\pi)</td>
</tr>
<tr>
<td>III</td>
<td>IV</td>
</tr>
</tbody>
</table>

Here is all this visually. Note that we had to add 180° to the angle measurement we got from the calculator (-11.3°) since the vector would terminate in the 2\(^{nd}\) quadrant if we were to start at \((0, 0)\). So we get 168.7°, which is the angle measurement from the positive x axis going counterclockwise.
168.7° from the positive x axis can also be described as 11.3° North of West (11.3° N of W, or W11.3°W), since the closest axis to the angle is the negative x axis (west) and we are going a little north of that:

(We saw a similar concept of this when we were working with bearings here in the Law of Sines and Cosines, and Areas of Triangles section).

Note that a vector that has a magnitude of 0 (and thus no direction) is called a zero vector. So hypothetically, the vector \( \overrightarrow{AA} \) would be a zero vector.

A unit vector is a vector with magnitude 1; in some applications, it’s easier to work with unit vectors. To find the unit vector that is associated with a vector (has same direction, but magnitude of 1), use the following formula: \( \hat{u} = \frac{u}{||u||} \) (just divide each component of a vector by its magnitude to get its unit vector). We’ll see some problems below.

**Vector Operations**

**Adding and Subtracting Vectors**

There are a couple of ways to **add** and **subtract** vectors. When we add vectors, geometrically, we just put the beginning point (initial point) of the second vector at the end point (terminal point) of the first vector, and see where we end up (new vector starts at beginning of one and
ends at end of the other). If the vectors aren't this way to begin with, we can move the second vector (as long as it has the same magnitude and direction, so it's like a slide) to be this way. You can think of adding vectors as connecting the diagonal of the parallelogram (a four-sided figure with two pairs of parallel sides) that contains the two vectors.

Do you see how when we add vectors geometrically, to get the sum, we can just add the \( x \) components of the vector, and the \( y \) components of the vectors?

![Diagram of vector addition and subtraction]

When we subtract two vectors, we just take the vector that's being subtracting, reverse the direction and add it to the first vector. This is because the negative of a vector is that vector with the same magnitude, but has an opposite direction (thus adding a vector and its negative results in a zero vector).

Note that to make a vector negative, you can just negate each of its components (\( x \) component and \( y \) component) (see graph below).

**Multiplying Vectors by a Number (Scalar)**

To multiply a vector by a number, or scalar, you simply stretch (or shrink if the absolute value of that number is less than 1), or you can simply multiply the \( x \) component and \( y \) component by that number. Notice also that the magnitude is multiplied by that scalar.

Do you see how two vectors that are parallel are just a multiple of each other? **Multiplying by a negative number** changes the direction of that vector.

Here's what subtracting vectors and also multiplying vectors by a scalar looks like:
Let's put all this together to perform the following vector operations, given the vectors shown:

\[ u: \langle 4, 1 \rangle = 4i + j \]
\[ v: \langle -2, 2 \rangle = -2i + 2j \]
\[ w: \langle 0, 6 \rangle = 6j \]
You may also see problems like this, where you have to tell whether the statement is true or false. Note that you want to look at where you end up in relation to where you started to see the resulting vector. If you end up exactly where you started from, the resulting vector is 0.

Here are a couple more examples of vector problems. Notice in the second set of problems when we are given a magnitude and direction of a vector, and have to find that vector, we use the following equation, like we did when we here in the Polar Coordinates, Equations and Graphs section: $\|v\|$ or the magnitude of a vector is like the “r” (radius) we saw for polar numbers: $\|(\text{Trigonometry})$ always seems to come back and haunt us! We’ll leave our answers in $ai + bj$ form.
### Applications of Vectors

Vectors are extremely important in many applications of science and engineering. Since vectors include both a length and a direction, many vector applications have to do with vehicle motion and direction.

We saw above that, given a magnitude and direction, we can find the vector with \( \| \vec{v} \| \) is the speed. This way we can add and subtract vectors, and get a resulting speed and direction for the new vector.

<table>
<thead>
<tr>
<th><strong>Vector Problems</strong></th>
<th><strong>Solutions</strong></th>
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<tbody>
<tr>
<td>For initial point ( \vec{P} ) and terminal point ( \vec{Q} ), write vector ( \vec{v} ) in from ( ai + bj ) (its position vector):</td>
<td><strong>Subtract initial point from terminal point:</strong></td>
</tr>
<tr>
<td>(a) ( P = (4, -2) ) ( Q = (0, 4) )</td>
<td>(a) ( \vec{v} = \overrightarrow{PQ} = (0 - 4, 4 - (-2)) = (-4, 6) = -4i + 6j )</td>
</tr>
<tr>
<td>(b) ( P = (0, 20) ) ( Q = (-1, 10) )</td>
<td>(b) ( \vec{v} = \overrightarrow{PQ} = (-1 - 0, 10 - 20) = (-1, -10) = -i - 10j )</td>
</tr>
<tr>
<td>Find the direction for (b).</td>
<td>For the direction of (-i - 10j), we have to use ( \tan^{-1} \left( \frac{y}{x} \right) ); we get 84.3° from the calculator, but this vector ends up in the third quadrant. We then have to add 180° to get an angle from the positive ( x ) axis of 264.3°.</td>
</tr>
<tr>
<td>Now let’s go the other way to get vectors back:</td>
<td>We’ll need to use ( \vec{v} = | \vec{v} | (\cos \alpha i + \sin \alpha j) ) to get our ( x ) and ( y ) components of the vector, respectively:</td>
</tr>
<tr>
<td>Find a vector ( \vec{v} ) in the form ( ai + bj ) given its magnitude and the angle it makes with the positive ( x ) axis:</td>
<td>(a) ( \vec{v} = | \vec{v} | (\cos \alpha i + \sin \alpha j) = 4 \cos(135°) i + 4 \sin(135°) j = )</td>
</tr>
<tr>
<td>(a) ( | \vec{v} | = 4, \ \alpha = 135° )</td>
<td>(a) ( = 4 \left( \frac{-\sqrt{2}}{2} \right) i + 4 \left( \frac{\sqrt{2}}{2} \right) j = -2\sqrt{2}i + 2\sqrt{2}j )</td>
</tr>
<tr>
<td>(b) ( | \vec{v} | = 2, \ \alpha = \frac{11\pi}{6} )</td>
<td>( \vec{v} = | \vec{v} | (\cos \alpha i + \sin \alpha j) = 2 \cos \left( \frac{11\pi}{6} \right) i + 2 \sin \left( \frac{11\pi}{6} \right) j )</td>
</tr>
<tr>
<td>(b) ( = 2 \left( \frac{\sqrt{3}}{2} \right) i + 2 \left( \frac{-1}{2} \right) j = \sqrt{3}i - j )</td>
<td></td>
</tr>
<tr>
<td>Find the unit vector having the same direction as ( \vec{v} ):</td>
<td>We’ll need to use ( \vec{u} = \frac{\vec{v}}{| \vec{v} |} ) to get unit vectors:</td>
</tr>
<tr>
<td>(a) ( \vec{v} = 6i )</td>
<td>(a) ( \vec{u} = \frac{\vec{v}}{| \vec{v} |} = \frac{6i}{\sqrt{6^2 + 0^2}} = \frac{6i}{6} = i )</td>
</tr>
<tr>
<td>(b) ( \vec{v} = 2i - j )</td>
<td>(b) ( \vec{u} = \frac{\vec{v}}{| \vec{v} |} = \frac{2i - j}{\sqrt{2^2 + (-1)^2}} = \frac{2i - j}{\sqrt{5}} = \frac{2}{\sqrt{5}}i - \frac{1}{\sqrt{5}}j = \frac{2\sqrt{5}}{5}i - \frac{\sqrt{5}}{5}j )</td>
</tr>
<tr>
<td>Note that we rationalized the fractions in (b).</td>
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</table>
Remember that a bearing (we saw here in the Law of Sines and Cosines, and Areas of Triangles section), is typically expressed a measure of the clockwise angle that starts due north or on the positive y axis (initial side) and terminates a certain number of degrees (terminal side) from that due north starting place. (This is also written, as in the case of a bearing of 40° as “40° east of north”, or “N40°E”).

(A lot of times, the bearing includes more directions, such as 70° west of north, also written as N70°W. In this case, the angle will start due north (straight up, or on the positive y axis) and go counterclockwise 70° (because it’s going west, or to the left, instead of east). Similarly, a bearing of 50° south of east, or E50°S, would be an angle that starts due east (on the positive x axis) and go clockwise 50° (towards the south, or down). Also, if you see a bearing of southwest, for example, the angle would be 45° south of west, or 225° clockwise from north, and so on.)

Each time a moving object changes course, you have to draw another line to the north to map its new bearing.

When there’s a tail wind, remember that you have to add this vector to the vector that the object is trying to go on (its programmed or “steered” course), to get the actual vector of the object. So remember:

\[
\text{ACTUAL COURSE} = \text{PROGRAMMED COURSE} + \text{COURSE of WIND}
\]
\[
\text{PROGRAMMED COURSE} = \text{ACTUAL COURSE} - \text{COURSE of WIND}
\]

Problem:
A plane is flying on a bearing of 25° south of west at 500 miles per hour (speed). Express the velocity of the plane (as a vector).

Solution:

<table>
<thead>
<tr>
<th>Picture</th>
<th>Math</th>
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<tbody>
<tr>
<td><img src="image" alt="vector" /></td>
<td>Let’s first draw the vector in the coordinate system to see how to get the clockwise angle from the positive x axis. 25° south of west means to go due west (negative x axis), and then go south (down) 25°. We can see that this angle is the same (co-terminal) of the angle that is 180° + 25° = 205° counter-clockwise from the positive x axis. We know that the speed is the same as the magnitude of the vector, so we can express the velocity vector in the form: ( \langle | \mathbf{v} | \cos \theta, | \mathbf{v} | \sin \theta \rangle ): ( \langle 500 \cos 205°, 500 \sin 205° \rangle = \langle -453.154, -211.309 \rangle ). Do you see how this coordinate point looks correct (3rd quadrant)?</td>
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Problem:
A sailboat is sailing on a bearing of 30° north of west at 6.5 miles per hour (in still water). A tail wind blowing 20 miles per hour in the direction 40° south of west alters the course of the boat.

Express the actual velocity of the sailboat as a vector. Then determine the actual speed and direction of the boat.

Solution:

Note that if we were given a vector for the actual course of the boat and had to come up with the vector for which the boat should be "steered", we would have to subtract the wind from the actual course.

Problem:
A cruise ship travels at a bearing of 40° at 60 mph for 3 hours, and changes course to a bearing of 120°. It then travels 40 mph for 2 hours. Find the distance the ship is from its original position and also its bearing from the original position.

Solution:
In this problem, Distance = rate \times time, since we are given rates and times and need to calculate distances.
Since no specific directions (like West of South) are given for these bearings, we will obtain the angles by measuring the clockwise angle that starts due north or on the positive y axis (East of North).
And remember that with a change of bearing, we have to draw another line to the north to map its new bearing.
Now that we have the angles, we can use vector addition to solve this problem; doing the problem with vectors is actually easier than using Law of Cosines:
Dot Product and Angle Between Two Vectors

The dot product of two vectors $v = ai + bj$ and $w = ci + dj$ (sort of like multiplying two vectors) is defined as $v \cdot w = ac + bd$; in other words, you multiply the two “x” parts of the vectors, and multiply the two “y” parts, and then add them together. The result is a scalar (single number).

Here is an example: if $v = -2i + 3j$ and $w = 2 + j$, the dot product $v \cdot w = (-2)(2) + (3)(1) = -1$.

We use dot products to find the angle measurements between two vectors; the cosine of the angle between two vectors is the dot product of the vectors, divided by the product of each of their magnitudes:

Note that we got 180 miles by multiplying rate x time to get distance, or 60 mph x 3 hours. Similarly, we get 80 miles by multiplying 40 mph by 2 hours.

The boat starts out at a 40° bearing (angle clockwise from the positive y axis, so the angle from the positive x axis is 90° – 40°, which is 50°). The second bearing is 120°, so the part of this angle underneath the x axis is 30° (120° – 90° = 30°). Now we can get the counterclockwise angle from the positive x axis by subtracting 30° from 360° to get 330°. We will use this to define the vector for this second bearing.

We use vector addition to get the new vector from the starting point to the ending point. Note again that we use angle measurements by going counterclockwise from the positive x axis:

$\langle 180\cos 50^\circ, 180\sin 50^\circ \rangle$
$+ \langle 80\cos 330^\circ, 80\sin 330^\circ \rangle$
$= \langle 184.984, 97.888 \rangle$

Do you see how this coordinate point looks correct (1st quadrant) with respect to the starting point? (New vector line is dashed).

To get the distance between the starting point and ending point, we have to take the magnitude of this new vector:

$\sqrt{184.984^2 + 97.888^2} = 209.287 \text{ miles.}$

To get the new bearing of the ship, we have to get the direction of the vector first: $\tan^{-1}\left(\frac{97.888}{184.984}\right) = 27.9^\circ$. (This is also the same as 27.9° North of East.)

To get the bearing from the positive y axis going clockwise, we have to subtract this from 90° to get 62.1°.

This is what we got when we did this problem using Law of Cosines!
\[
\cos \theta = \frac{u \cdot v}{\|u\| \|v\|}; \quad \theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\| \|v\|}\right)
\]

(And we don’t need to worry about getting the correct quadrant when putting this in the calculator!)

So we might be able to use this formula instead of, say, the Law of Cosines, for applications. Note that if the dot product of two vectors is 0, the vectors form right angles, or are orthogonal, since the \(\cos\) of 90° is 0 (and thus the whole expression will be 0).

And remember that we noted above that if two vectors are parallel, then one is a “multiple” of another, or \(v = aw\). So for example, the vector \(v = -2i + 3j\) would be parallel to the vector \(v = -4i + 6j\). If vectors are parallel, the angle between them is either 0 (if they are the same vector) or \(\pi\).

Here are some example problems:
3D Vectors – Vectors in Space

We've been dealing with vectors (and everything else!) in the two dimensional plane, but "real life" is actually three dimensional, so we need to know how to work in 3D, or space, too. A 3D coordinate system is typically drawn like this, with the z axis going "up". Note that the positive x axis comes forward, and the positive y axis to the right of that. One way to envision this is if you hold your right hand out pointing with the thumb up, the thumb and the first two fingers point to the positive z, x, and y axes; this is called the Right Hand Rule:

Find the dot product v•w and the angle between the two vectors.

Indicate if the vectors are orthogonal, parallel, or neither.

(a) v = 4i - 2j; w = -2i + j
(b) v = 3i - j; w = -i + 2j
(c) v = -i + 3j; w = 6i + 2j
(d) v = -i; w = -2i + 2j

Picture for (c) above:

\[ \mathbf{v} \cdot \mathbf{w} = (4)(-2) + (-2)(1) = -10 \]

Since \( \langle 4, -2 \rangle = -2\langle -2, 1 \rangle \), the vectors are parallel.

\[ \theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} \right) = \cos^{-1} \left( \frac{-10}{\sqrt{22} \sqrt{10}} \right) = \cos^{-1} \left( -\frac{1}{\sqrt{2}} \right) = 135^\circ \]

(b) \( v \cdot w = (3)(-1) + (-1)(2) = -5 \)

The vectors are neither orthogonal nor parallel.

\[ \theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} \right) = \cos^{-1} \left( \frac{-5}{\sqrt{10} \sqrt{10}} \right) = \cos^{-1} \left( -\frac{1}{\sqrt{2}} \right) = 135^\circ \]

(c) \( v \cdot w = (-1)(6) + (3)(2) = 0 \)

Since \( v \cdot w = 0 \), the vectors are orthogonal (see picture to left).

\[ \theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} \right) = \cos^{-1} \left( \frac{0}{\sqrt{50}} \right) = \cos^{-1} \left( 0 \right) = 90^\circ \]

(d) \( v \cdot w = (-1)(-2) + (0)(2) = 2 \)

The vectors are neither orthogonal nor parallel.

\[ \theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} \right) = \cos^{-1} \left( \frac{2}{\sqrt{5} \sqrt{2}} \right) = \cos^{-1} \left( \frac{\sqrt{2}}{\sqrt{10}} \right) = 45^\circ \]

Find x so that the following vectors are orthogonal:

(a) \( v = 4i - 2j; w = xi + j \)
(b) \( v = j; w = 3i + xj \)

We'll need to find \( v \cdot w \) and make sure it equals 0:

(a) \( v \cdot w = (4)(x) + (-2)(1) = 0; 4x = 2; x = \frac{1}{2} \)
(b) \( v \cdot w = (0)(3) + (1)(x) = 0; x = 0 \)
**Geometric Vectors** in 3D are still directed line segments, but in the \(xyz\)-plane. We still can find the vector between two coordinate points by “subtracting” the first vector from the second.

So, for example, if \(A\) is \((-4, 2, 7)\) and \(B\) is \((-3, 8, 0)\), the vector \(\overrightarrow{AB}\) is **second point minus first point**, or \((-3 - (-4), 8 - 2, 0 - 7) = (1, 6, -7)\). Note also that vectors can also be written in the form \(ai + bj + ck\), so this vector can also be written as \(i + 6j - 7k\).

The **magnitude** of the 3D vector, written \(\|\overrightarrow{AB}\|\) is still the **distance** between the two points (like taking hypotenuse of a right triangle twice actually), or \(\sqrt{x^2 + y^2 + z^2}\). So for our points \(A\) and \(B\) above, \(\|\overrightarrow{AB}\| = \sqrt{1^2 + 6^2 + (-7)^2} = \sqrt{86}\).

**Vector Operations in Three Dimensions**

Adding, subtracting 3D vectors, and multiplying 3D vectors by a scalar are done the same way as 2D vectors; you just have to work with **three components**.

Like for 2D vectors, the **dot product** of two vectors \(v = ai + bj + ck\) and \(w = di + ej + fk\) (sort of like multiplying two vectors) is defined as \(v \cdot w = ad + be + cf\); in other words, you multiply the two “\(x\)” parts of the vectors, multiply the two “\(y\)” parts, multiply the two “\(z\)” parts, and then add them together. The result is a scalar (single number).

Again, like for 2D, we use dot products to find the **angle measurements between two vectors**; the cosine of the angle between two vectors is the dot product of the vectors, divided by the product of each of their magnitudes:

\[
\cos \theta = \frac{u \cdot v}{\|u\| \|v\|}; \quad \theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\| \|v\|}\right)
\]
<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
</table>
| (a) Find the equation of the line that passes through the points (2,−3,1) and (4,0,−2). Write down all three forms of this equation. | (a) We first need to find the vector \( \mathbf{v} \) that is parallel to the line that we are trying to find. To get this, we find the vector between the two points: \( \langle 4−2,0−(−3),−2−1 \rangle = \langle 2,3,−3 \rangle. \) To get the vector form of the line, we can use either point as the initial point (we’ll use first one) to get: \( \langle 2,−3,1 \rangle + t \langle 2,3,−3 \rangle, \) or \( 2 + 2t \), \( −3 + 3t \), \( 1 − 3t \). \[
x = 2 + 2t \\
y = −3 + 3t \\
\frac{z−1}{−3} = \frac{(1−z)}{3}
\] The other forms are \( y = −3 + 3t \) and (solving for \( t \)): \[
\frac{x−2}{2} = \frac{y + 3}{3} = \frac{z−1}{−3} \left( \frac{1−z}{3} \right).
\]
(b) If this line passes through the XY plane, give the coordinate of the point of intersection. | (b) To get the point where this line passes through the XY plane, we can set \( z \) to 0 to get \( t \) in this equation: \( z = 1 − 3t; \) \( 0 = 1 − 3t; \) \( t = \frac{1}{3} \), and then put this value of \( t \) in the line’s equation: \( \langle 2,−3,1 \rangle + t \langle 2,3,−3 \rangle; \) \( \langle 2,−3,1 \rangle + \left( \frac{1}{3} \right) \langle 2,3,−3 \rangle = \langle \frac{2}{3},−2,0 \rangle. \) So this point will be \( \left( \frac{2}{3},−2,0 \right) \). |
| Find the equation of the plane that passes through the point (3,−3,1), and contains the line \[
\frac{2−x}{3} = \frac{y + 1}{4} = \frac{z}{2}.
\] | We know the point \( (3,−3,−1) \) is on the plane (given). Let’s rewrite the line equation in vector form \( \langle x,y,z \rangle = \langle x_0,y_0,z_0 \rangle + t \langle a,b,c \rangle \): \( \langle x,y,z \rangle = \langle −2,1,0 \rangle + t \langle −3,4,2 \rangle \) (we have to set each expression to \( t \), and solve back for \( x \), \( y \), and \( z \). Also note that we had to multiply the \( x \) expression by −1). So then we know the points \( (−2,1,0) \) (when \( t = 0 \)) and \( (−5,5,2) \) (when \( t = 1 \)) are also on the plane. Then we can use the method above (3D Vector Problem) to find the equation of the plane, given three points. |
| Find the equation of the plane that contains the lines \[
\frac{2−x}{3} = \frac{y + 1}{4} = \frac{z}{2}
\] and \[
\frac{x}{2} = \frac{y − 2}{3} = \frac{z − 3}{−2}.
\] | This one’s a little easier, since we can just find the cross product of the two vectors to get the vector that is normal, or perpendicular to the plane. To do this, we rewrite the equations in vector form: \( \langle x,y,z \rangle = \langle −2,1,0 \rangle + t \langle −3,4,2 \rangle \) and \( \langle x,y,z \rangle = \langle 0,−2,−3 \rangle + t \langle 2,3,−2 \rangle. \) \[
\begin{vmatrix}
  i & j & k \\
  2 & 3 & −2 \\
  −3 & 4 & 2
\end{vmatrix} = \begin{vmatrix}
  4 & 2 & 3 \\
  −3 & 2 & −2 \\
  3 & 4 & k
\end{vmatrix} = (−8−6)i−(6−4)j+(−9−8)k \\
=−14i−2j−17k.
\] Again, we know that the equation of the plane where \( \langle a,b,c \rangle \) is perpendicular at a certain point \( (x_0,y_0,z_0) \) is \( ax + by + cz = d \), where \( d = ax_0 + by_0 + cz_0 \). Since the vector we’re dealing with is \( \langle 14,−2,−17 \rangle \), we have \( 14x−2y−17z = d \). To get \( d \), we plug in any point — let’s use \( (−2,1,0) \) (when \( t = 0 \) on the first line) — for \( x, y, \) and \( z \): \( d = 14(−2)−2(1)−17(0) = −30 \). So the equation of this plane is \( 14x−2y−17z = −30 \). |
Here are more problems that look more difficult, but are actually easier, since we’re finding a plane perpendicular to the vector or line:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
</table>
| Find the equation of the plane that passes through the point \((4,0,-1)\) and is perpendicular to the vector \(n = i - 3j + 2k\). | Remember that the equation of the plane for a vector perpendicular to a certain plane at point \((x_0, y_0, z_0)\) is \(ax + by + cz = d\), where \(d = ax_0 + by_0 + cz_0\) (plug in the point to get \(d\)).
   So to get the equation of the plane that is perpendicular to the vector \(\langle 1, -3, 2 \rangle\), we have \(x - 3y + 2z = d\).
   One way to get \(d\) is to set \(x - 3y + 2z\) to 0, and then put in “\(x - 4\)”, “\(y\)”, and “\(z + 1\)” (from the point \((4,0,-1)\)) for \(x\), \(y\), and \(z\), respectively:
   \((x - 4) - 3(y) + 2(z + 1) = 0\), or \(x - 4 - 3y + 2z + 2 = 0\).
   So the equation of this plane is \(x - 3y + 2z = 2\). |
| Find the equation of the plane that passes through the point \((3,-3,1)\), and is perpendicular to the line \(\frac{2 - x}{3} = \frac{y + 1}{4} = \frac{z}{2}\). | We know the direction vector of this line is \(\langle -3, 4, 2 \rangle\) (we have to set each expression to \(t\), and solve back for \(x, y\), and \(z\). Also note that we had to multiply the \(x\) expression by \(-1\)).
   Again, we know that the equation of the plane for a vector perpendicular to a certain plane at point \((x_0, y_0, z_0)\) is \(ax + by + cz = d\), where \(d = ax_0 + by_0 + cz_0\) (plug in the point to get \(d\)).
   So to get the equation of the plane that is perpendicular to the vector \(\langle -3, 4, 2 \rangle\), we have \(-3x + 4y + 2z = d\).
   One way to get \(d\) is to set \(-3x + 4y + 2z\) to 0, and then put in “\(x - 3\)”, “\(y + 3\)”, and “\(z - 1\)” (from the point \((3,-3,1)\)) for \(x\), \(y\), and \(z\), respectively:
   \(-3(x - 3) + 4(y + 3) + 2(z - 1) = 0\) or \(-3x + 9 + 4y + 12 + 2z - 2 = 0\).
   So the equation of this plane is \(-3x + 4y + 2z = -19\). |